$\nu = 0.2$, $k\nu' = 0.3$, $\gamma = 6$, $\eta_0 = 0.067$, $\operatorname{tg}\varphi = 0.8$. The values $x' = (x \operatorname{tg}\varphi)^{-1}$ ($x_0 = \operatorname{ctg}\varphi$, $x \geqslant x_0$) are plotted along the abscissa.

The numerical results obtained show that the second approximation of the temperature problem of the deformation of a thin-walled conical pipe provides very high accuracy even at large aperture angles of the pipe φ (for example when $\operatorname{tg}\varphi = 0.8$). The first (asymptotic) approximation describes the stress state of a thin-walled pipe with small aperture angle with sufficient accuracy.

In the case of thick-walled pipes, the second approximation is found to provide sufficient accuracy.

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THE EFFECTIVE CHARACTERISTICS OF PIEZOACTIVE COMPOSITES WITH CYLINDRICAL INCLUSIONS†

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Using the averaging method [1, 2], a procedure is proposed for determining accurate values of the effective moduli of elasticity, piezoelectric moduli and permittivities of piezoactive composites of periodic structure with unidirectional fibres having the form of a circular cylinder. The accurate values are obtained by the analytical solution of the problems in a periodicity cell.

THE AVERAGING method has previously been used to determine the effective properties of layered piezoelectric composites in [3, 4]. To investigate the effect of the properties of fibre piezoelectric composites approximate formulas have been proposed based on a statistical approach [5] and on the method of matching and variational estimates [6].

1. Consider the non-homogeneous problem of the theory of electro-elasticity for a piezoactive composite with a periodic structure. It is described by the following system of equations [7] and boundary conditions:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{F} = 0, \quad \nabla \cdot \mathbf{D} = 0$$

$$\boldsymbol{\sigma} = \mathbf{C}\left(\frac{x}{\varepsilon}\right) \cdot\cdot \nabla \mathbf{u} + \mathbf{e}^T \left(\frac{x}{\varepsilon}\right) \cdot \nabla \varphi \tag{1.1}$$

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$$\mathbf{D}=\mathbf{e}\left(\frac{x}{\varepsilon}\right)\cdot\cdot\nabla\mathbf{u}-\mathscr{E}\left(\frac{x}{\varepsilon}\right)\cdot\nabla\varphi$$
$$\mathbf{u}|_{S_1}=\mathbf{u}^0,\quad \sigma\cdot\mathbf{n}|_{S_2}=\sigma^0$$
$$\varphi|_{S_3}=\varphi^0,\quad \mathbf{D}\cdot\mathbf{n}|_{S_4}=\varkappa^0 \tag{1.2}$$

Here $\mathbf{u}$ is the displacement vector, φ is the electric potential, $\mathbf{F}$ is the vector of volume forces, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{D}$ is the electric-induction vector, $\mathbf{C}$ is the tensor of the moduli of elasticity with components C_{ijkl}, $\mathbf{e}$ is the tensor of the piezoelectric moduli with components e_{kij}, $\mathscr{E}$ is the permittivity tensor with components $\mathscr{E}_{kn}$, $\mathbf{n}$ is the unit vector of the external normal to the boundary S of the region occupied by the composite, S_1 is the part of the boundary S on which the displacements are specified, $S_2 = S\backslash S_1$ is the part of the boundary on which the stresses are specified, S_3 is the part of the boundary S coated with electrodes on which the potential is specified, and $S_4 = S\backslash S_3$ is the part of the boundary free from electrode while ε defines the characteristic linear dimensions of the periodicity cell relative to the linear dimensions of the composite.

We will associate with the periodicity cell a local system of coordinates $\xi(\xi_1, \xi_2, \xi_3)$, $\xi_k = x_k/\varepsilon$. Then using the average method [1, 2] the solution of problem (1.1), (1.2) will be sought in the form of the expansions

$$\mathbf{u}=\mathbf{u}_0(x)+\varepsilon\mathbf{u}_1(x,\xi)+\varepsilon^2\mathbf{u}_2(x,\xi)+\ldots$$
$$\varphi=\varphi_0(x)+\varepsilon\varphi_1(x,\xi)+\varepsilon^2\varphi_2(x,\xi)+\ldots \tag{1.3}$$

substitution of which into Eq. (1.1), taking into account the equation $\nabla = \nabla_x + \varepsilon^{-1}\nabla_\xi$, where ∇_x, ∇_ξ are Nabla-operators in the system x and ξ, respectively, leads to the sequence of equations (δ_{ik} is the Kronecker delta)

$$\nabla_x\cdot\boldsymbol{\sigma}_{k-1}+\nabla_\xi\cdot\boldsymbol{\sigma}_k+\delta_{1k}\mathbf{F}=0$$
$$\nabla_x\cdot\mathbf{D}_{k-1}+\nabla_\xi\cdot\mathbf{D}_k=0,\quad k=0,1,2,\ldots \tag{1.4}$$

Here

$$\boldsymbol{\sigma}_k=\mathbf{C}(\xi)\cdot\cdot(\nabla_x\mathbf{u}_k+\nabla_\xi\mathbf{u}_{k+1})+\mathbf{e}^T(\xi)\cdot(\nabla_x\varphi_k+\nabla_\xi\varphi_{k+1})$$
$$\mathbf{D}_k=\mathbf{e}(\xi)\cdot\cdot(\nabla_x\mathbf{u}_k+\nabla_\xi\mathbf{u}_{k+1})-\mathscr{E}(\xi)\cdot(\nabla_x\varphi_k+\nabla_\xi\varphi_{k+1}) \tag{1.5}$$

are the corresponding components in the expansion in terms of the parameter ε of the stress tensor $\boldsymbol{\sigma}$ and the electric-induction vector $\mathbf{D}$. In (1.4) the components with a negative index must be taken to be equal to zero.

Note that $\mathbf{u}_k(x, \xi)$, φ_k, $\delta_k(x, \xi)$, $\mathbf{D}_k(x, \xi)$, $(k \geqslant 1)$ are the functions that are periodic with respect to the fast coordinate ξ.

The problem at the zeroth stage ($k = 0$) (we will call it the problem in a cell) can be represented in the form

$$\nabla_\xi\cdot\boldsymbol{\sigma}_0=0,\quad \nabla_\xi\cdot\mathbf{D}_0=0 \tag{1.6}$$

and consists of finding periodic functions u_1, φ_1 in the periodicity cell. To do this we will represent u_1 and φ_1 in the form

$$\mathbf{u}_1(x,\xi)=\mathbf{N}(\xi)\cdot\cdot\nabla_x\mathbf{u}_0+\mathbf{R}(\xi)\cdot\nabla_x\varphi_0$$
$$\varphi_1(x,\xi)=\mathbf{S}(\xi)\cdot\cdot\nabla_x\mathbf{u}_0+\boldsymbol{\Phi}(\xi)\cdot\nabla_x\varphi_0 \tag{1.7}$$

where $\mathbf{N}(\xi)$, $\mathbf{R}(\xi)$, $\mathbf{S}(\xi)$, $\boldsymbol{\Phi}(\xi)$ are tensor functions, periodic in ξ, of the third, second, second and first ranks, respectively.

Using relations (1.5) we can write the following representations:

$$\boldsymbol{\sigma}_0=(\mathbf{C}+\mathbf{C}\cdot\cdot\nabla_\xi\mathbf{N}+\mathbf{e}^T\cdot\nabla_\xi\mathbf{S})\cdot\cdot\nabla_x\mathbf{u}_0+(\mathbf{e}^T+\mathbf{C}\cdot\cdot\nabla_\xi\mathbf{R}+\mathbf{e}^T\cdot\nabla_\xi\boldsymbol{\Phi})\cdot\nabla_x\varphi_0$$
$$\mathbf{D}_0=(\mathbf{e}+\mathbf{e}\cdot\cdot\nabla_\xi\mathbf{N}-\mathscr{E}\cdot\nabla_\xi\mathbf{S})\cdot\cdot\nabla_x\mathbf{u}_0-(\mathscr{E}-\mathbf{e}\cdot\cdot\nabla_\xi\mathbf{R}+\mathscr{E}\nabla_\xi\boldsymbol{\Phi})\cdot\nabla_x\varphi_0 \tag{1.8}$$

Equations (1.6) will then be satisfied for any $\mathbf{u}_0(x)$, $\varphi_0(x)$ if the following two groups of equalities hold:

$$\begin{aligned}&\nabla_\xi\cdot(\mathbf{C}\cdot\cdot\nabla_\xi\mathbf{N}+\mathbf{e}^{\tau}\cdot\nabla_\xi\mathbf{S}+\mathbf{C})=0\\&\nabla_\xi\cdot(\mathbf{e}\cdot\cdot\nabla_\xi\mathbf{N}-\mathscr{E}\cdot\nabla_\xi\mathbf{S}+\mathbf{e})=0\end{aligned}\tag{1.9}$$

$$\nabla_\xi\cdot(\mathbf{C}\cdot\cdot\nabla_\xi\mathbf{R}+\mathbf{e}^{\tau}\cdot\nabla_\xi\mathbf{\Phi}+\mathbf{e}^{\tau})=0$$

$$\nabla_\xi\cdot(\mathbf{e}\cdot\cdot\nabla_\xi\mathbf{R}-\mathscr{E}\cdot\nabla_\xi\mathbf{\Phi}-\mathscr{E})=0\tag{1.10}$$

Hence, the problem in a cell has been reduced to determining periodic local functions $N_{kpq}(\xi)$, $S_{pq}(\xi)$ from the system of equations (1.9)

$$\begin{aligned}&(C_{ijkl}N_{kpq,l}+e_{kij}S_{pq,k})_{,j}=-C_{ijpq,j}\\&(e_{kij}N_{ipq,j}-\mathscr{E}_{kn}S_{pq,n})_{,k}=-e_{kpq,k}\end{aligned}\tag{1.11}$$

and the functions $R_{kq}(\xi)$, $\Phi_q(\xi)$ from system (1.10)

$$(C_{ijkl}R_{kq,l}+e_{kij}\Phi_{q,k})_{,j}=-e_{qij,j},\quad (e_{kij}R_{iq,j}-\mathscr{E}_{kn}\Phi_{q,n})_{,k}=\mathscr{E}_{kq,k}\tag{1.12}$$

in the periodicity cell.

For Eqs (1.11) and (1.12) to be solvable we must require that the following condition is satisfied:

$$\langle N_{kpq}(\xi)\rangle=\langle S_{pq}(\xi)\rangle=\langle R_{pq}(\xi)\rangle=\langle\Phi_q(\xi)\rangle=0\tag{1.13}$$

where the angle brackets denote the average over the volume of the periodicity cell. In addition to conditions (1.13), it is necessary to add to (1.11) and (1.12) the conditions of continuity at the boundary of a fibre and of the displacement matrix, the stress vector, the electric potential and the normal component of the electric-induction vector

$$[N_{kpq}]=0,\quad [S_{pq}]=0\tag{1.14}$$

$$\begin{aligned}&[C_{ijkl}N_{kpq,l}+e_{kij}S_{pq,k}+C_{ijpq}]n_j=0\\&[e_{kij}N_{ipq,j}-\mathscr{E}_{kn}S_{pq,n}+e_{kpq}]n_k=0\\&[R_{kq}]=0,\ [G_{ijkl}R_{kq,l}+e_{kij}\Phi_{q,k}+e_{qij}]n_j=0\\&[\Phi_q]=0,\ [e_{kij}R_{iq,j}-\mathscr{E}_{kn}\Phi_{q,n}-\mathscr{E}_{kq}]n_k=0\end{aligned}\tag{1.15}$$

where the square brackets denote jumps in the values of the quantities included in them at the interface between phases, and n_j are the components of the unit vector of the external normal to the fibre surface.

After solving problems (1.11) and (1.12) the effective moduli of the composite can be obtained. To do this we consider the system of equations (1.4) with $k=1$ and we integrate them over the volume of the periodicity cell

$$\nabla_x\langle\boldsymbol{\sigma}_0\rangle+\langle\nabla_\xi\cdot\boldsymbol{\sigma}_1\rangle+\mathbf{F}=0,\quad \nabla_x\cdot\langle\mathbf{D}_0\rangle+\langle\nabla_\xi\cdot\mathbf{D}_1\rangle=0\tag{1.16}$$

The second terms in (1.16) vanish by virtue of Gauss' theorem and the conditions of periodicity, and hence the macroscopic equations of equilibrium have the form

$$\nabla_x\cdot\boldsymbol{\sigma}^*+\mathbf{F}=0,\quad \nabla_x\cdot\mathbf{D}^*=0\tag{1.17}$$

where $\boldsymbol{\sigma}^*=\langle\boldsymbol{\sigma}_0\rangle$, $\mathbf{D}^*=\langle\mathbf{D}_0\rangle$. By comparing this with relations (1.8) we can determine the effective properties of the piezoactive composite

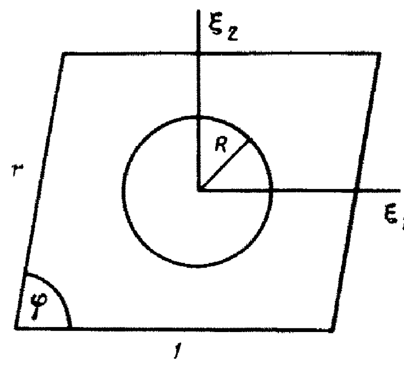

FIG. 1.

$$C^*_{ijpq} = \langle C_{ijpq} + C_{ijkl}N_{kpq,l} + e_{kij}S_{pq,k}\rangle \tag{1.18}$$

$$\mathscr{E}_{kn}^* = \langle \mathscr{E}_{kn} + \mathscr{E}_{kl}\Phi_{n,l} - e_{kij}R_{in,j}\rangle$$

$$e^*_{kij} = \langle e_{kij} + e_{kmn}N_{mij,n} - \mathscr{E}_{kn}S_{ij,n}\rangle$$

2. We will consider a piezoactive composite with a periodic structure having unidirectional fibres in the form of a circular cylinder. The x_3 axis of a rectangular Cartesian system of coordinates $x(x_1, x_2, x_3)$ is chosen to coincide with the direction of the fibres. Henceforth we will assume that the section of the periodicity cell by a plane x_1x_2 is a parallelogram, the length of one of the sides of which is unity, while the length of the other is r (Fig. 1). When $r = 1$, $\varphi = \pi/2$ we have a square structure while the case $r = 1$, $\varphi = \pi/3$ corresponds to a hexagonal structure. For a square structure the maximum volume concentration of inclusions γ is 0.78 of the volume of the composite, while for the hexagonal structure it is 0.86 of the volume of the composite.

We will assume that the components of the piezoelectric composite belong to symmetry class $6mm$, while the polarization axis coincides with the directions of the fibres. The defining relations from (1.1) can then be represented in the form [7]

$$\begin{aligned}
&\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} - e_{31}E_3,\\
&\sigma_{22} = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} - e_{31}E_3\\
&\sigma_{33} = C_{13}(\varepsilon_{11} + \varepsilon_{22}) + C_{33}\varepsilon_{33} - e_{33}E_3,\\
&\sigma_{\alpha 3} = 2C_{44}\varepsilon_{\alpha 3} - e_{15}E_\alpha, \quad \sigma_{12} = 2C_{66}\varepsilon_{12}\\
&D_\alpha = 2e_{15}\varepsilon_{\alpha 3} + \mathscr{E}_{11}E_\alpha, \quad \alpha = 1, 2,\\
&D_3 = e_{31}(\varepsilon_{11} + \varepsilon_{22}) + e_{33}\varepsilon_{33} + \mathscr{E}_{33}E_3
\end{aligned} \tag{2.1}$$

Here we have used the traditional double-index notation of the components of the tensors C_{ijkl} and e_{klm}, according to which the symmetrical pair of indices is replaced by 1 according to the following rule: ij is replaced by i when $i = j$ and by $9 - i - j$ when $i \neq j$.

If the fibres are made of a piezoactive material, while the binding matrix is a passive dielectric ($e_{klm} \equiv 0$), the piezoelectric composite has the structure 1–3 according to the classification in [8]. Otherwise, it has a structure 3–1, which includes, in particular, porous ceramics.

Composites of this structure are widely used in manufacturing piezoelectric transducers with high volume piezoelectric sensitivity, low impedance and a uniform amplitude-frequency characteristic [8, 9].

For a composite of the chosen structure the unknown functions N_{kpq}, S_{pq}, R_{kq}, Φ_q depend on two coordinates ξ_1, ξ_2 and are found from problems (1.11), (1.14) and (1.12) and (1.15), which we denote by I_{pq} and J_q, respectively. In view of the symmetry of C_{ijpq}, e_{kpq} with respect to the last two indices it is sufficient to solve six problems I_{pq} for $p \leq q$ and three problems J_q. For brevity, in what follows we will omit the last pair of indices on the functions N_{kpq}, S_{pq} and the last index on the functions R_{kq}, Φ_q.

From the solution of the problems $I_{\beta\beta}(p = q = \beta = 1, 2, 3)$, which, taking relations (2.1) into account, we will write in the form

$$\begin{gathered}(C_{11kl}N_{k,l}+C_{1\beta})_{,1}+(C_{12kl}N_{k,l})_{,2}=0\\(C_{21kl}N_{k,l})_{,1}+(C_{22kl}N_{k,l}+C_{2\beta})_{,2}=0\\N_3=S\equiv0,\ \ k,\ l=1,\ 2\end{gathered} \tag{2.2}$$

we can find, using (1.18), the effective values of the non-zero parameters:

$$\begin{gathered}C_{1\beta}{}^*=\langle C_{1\beta}+C_{11}N_{1,1}+C_{12}N_{2,2}\rangle\\C_{2\beta}{}^*=\langle C_{2\beta}+C_{12}N_{1,1}+C_{11}N_{2,2}\rangle\\C_{3\beta}{}^*=\langle C_{3\beta}+C_{13}N_{1,1}+C_{13}N_{2,2}\rangle\\C_{6\beta}{}^*=\langle C_{66}(N_{1,2}+N_{2,1})\rangle\\e_{3\beta}{}^*=\langle e_{3\beta}+e_{31}N_{1,1}+e_{31}N_{2,2}\rangle\end{gathered} \tag{2.3}$$

The solution of problem I_{12}

$$\begin{gathered}(C_{11kl}N_{k,l})_{,1}+(C_{12kl}N_{k,l}+C_{66})_{,2}=0\\(C_{21kl}N_{k,l}+C_{66})_{,1}+(C_{22kl}N_{k,l})_{,2}=0\end{gathered} \tag{2.4}$$

enables us to determine

$$C_{66}{}^*=\langle C_{66}+C_{66}(N_{1,2}+N_{2,1})\rangle,\qquad e_{36}{}^*=\langle e_{31}(N_{1,1}+N_{2,2})\rangle \tag{2.5}$$

By solving problems $I_{\alpha3}$ ($\alpha = 1, 2$)

$$\begin{gathered}(C_{44}N_{3,1}+e_{15}S_{,1}+\delta_{1\alpha}C_{44})_{,1}+(C_{44}N_{3,2}+e_{15}S_{,2}+\delta_{2\alpha}C_{44})_{,2}=0\\(e_{15}N_{31}-\mathscr{E}_{11}S_{,1}+\delta_{1\alpha}e_{15})_{,1}+(e_{15}N_{32}-\mathscr{E}_{11}S_{,2}+\delta_{2\alpha}e_{15})_{,2}=0,\ \ N_1=N_2\equiv0\end{gathered} \tag{2.6}$$

we can find the following effective parameters:

$$\begin{gathered}C_{55}{}^*=\langle C_{44}+C_{44}N_{3,1}+e_{15}S_{,1}\rangle\\C_{45}{}^*=\langle C_{44}N_{32}+e_{15}S_{,2}\rangle\\e_{15}{}^*=\langle e_{15}+e_{15}N_{3,1}-\mathscr{E}_{11}S_{,1}\rangle\\e_{25}{}^*=\langle e_{15}N_{32}-\mathscr{E}_{11}S_{,2}\rangle\ \ (\alpha=1)\\C_{44}{}^*=\langle C_{44}+C_{44}N_{3,2}+e_{15}S_{,2}\rangle\\e_{14}{}^*=\langle e_{15}N_{3,1}-\mathscr{E}_{11}S_{,1}\rangle\\e_{24}{}^*=\langle e_{15}+e_{15}N_{3,2}-\mathscr{E}_{11}S_{,2}\rangle\ \ (\alpha=2)\end{gathered} \tag{2.7}$$

From the solution of the problems J_α ($\alpha = 1, 2$), which are identical in form with Eqs (2.6), in which the functions N_3 and S are replaced by R_3 and Φ, while the coefficients on the Kronecker symbols C_{44} and e_{15} are replaced by e_{15} and $-\mathscr{E}_{11}$ respectively, we find

$$\begin{gathered}\mathscr{E}_{11}{}^*=\langle\mathscr{E}_{11}+\mathscr{E}_{11}\Phi_{,1}-e_{15}R_{3,1}\rangle\\\mathscr{E}_{12}{}^*=\langle\mathscr{E}_{11}\Phi_{,2}-e_{15}R_{3,2}\rangle\ \ (\alpha=1)\\\mathscr{E}_{22}{}^*=\langle\mathscr{E}_{11}+\mathscr{E}_{11}\Phi_{,2}-e_{15}R_{3,2}\rangle\ \ (\alpha=2)\end{gathered} \tag{2.8}$$

We can find the effective permittivity $\mathscr{E}_{33}^*$ from the formula

$$\mathscr{E}_{33}^* = \langle \mathscr{E}_{33} - e_{31} R_{1,1} - e_{31} R_{2,2} \rangle \tag{2.9}$$

where the functions R_1 and R_2 are the solutions of problem J_3, which is obtained from (2.2) by replacing N_k by R_k, S by Φ and $C_{1\beta}$, $C_{2\beta}$ by e_{31}.

3. To solve the problems I_{pq} and J_q we will use the methods of the theory of functions of a complex variable [10]. For problems $I_{\beta\beta}$, I_{12}, J_3 the solutions can be expressed in terms of the functions

$$\varphi_1(z) = \frac{a_0}{R} z + \sum_{k=1}^{\infty}{}^{*} a_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}$$

$$\psi_1(z) = \frac{b_0}{R} z + \sum_{k=1}^{\infty}{}^{*} b_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!} + \sum_{k=1}^{\infty}{}^{*} a_k R^k \frac{Q^{(k-1)}(z)}{(k-1)!} \tag{3.1}$$

$$\varphi_2(z) = \sum_{k=1}^{\infty}{}^{*} C_k \left(\frac{z}{R}\right)^k, \quad \psi_2(z) = \sum_{k=1}^{\infty}{}^{*} d_k \left(\frac{z}{R}\right)^k$$

where $z = \xi_1 + i\xi_2$, $\zeta(z)$ is Weierstrass' zeta-function, $Q(z)$ is Natanzon's function [10], a_0, b_0, a_k, b_k, C_k, d_k are certain complex constants, and Σ^* represents summation over odd values of the index. Here $\varphi_1(z)$, $\psi_1(z)$ are functions which relate to the region of the binding material, while $\varphi_2(z)$ and $\psi_2(z)$ relate to the region of the fibre containing the origin of coordinates (Fig. 1).

The coefficients a_0 and b_0 are expressed in terms of a_1 and b_1 from the condition for the required functions to be periodic [11]. The coefficients a_k, b_k, C_k and d_k ($k \geqslant 1$) are determined from the infinite system of linear algebraic equations obtained by satisfying the matching conditions on the interface between the media (1.14) and (1.15). This infinite system belongs to the normal type and its form is known [11].

After solving the linear system of algebraic equations, the effective characteristics of the piezoelectric composite can be found in terms of the coefficients a_1 and C_1 from the formulas

$$C_{1\beta}^* = \langle C_{1\beta} \rangle - \gamma \operatorname{Re} \{A_1 + p_2(1) A_2 + p_1(\beta)\}$$
$$C_{2\beta}^* = \langle C_{2\beta} \rangle + \gamma \operatorname{Re} \{A_1 - p_2(1) A_2 + p_1(\beta)\}$$
$$C_{3\beta}^* = \langle C_{3\beta} \rangle - \gamma [C_{31}] \operatorname{Re} A_2, C_{6\beta}^* = -\gamma \operatorname{Im} A_1 \tag{3.2}$$

$$e_{3\beta}^* = \langle e_{3\beta} \rangle - \gamma [e_{31}] \operatorname{Re} A_2, C_{66}^* = C_{66}^1 - \gamma \operatorname{Im} A_1$$
$$e_{36}^* = -\gamma [e_{31}] \operatorname{Re} A_2, \mathscr{E}_{33}^* = \langle \mathscr{E}_{33} \rangle + \gamma [e_{31}] \operatorname{Re} A_2$$
$$p_1(\beta) = ([C_{2\beta}] - [C_{1\beta}])/2, \; p_2(\beta) = ([C_{1\beta}] + [C_{2\beta}])/2$$
$$A_1 = (1 + \varkappa_1) a_1/R, \; A_2 = (\varkappa_2 C_1 - \bar{C}_1)/C_{66}^2/R, \quad \varkappa_\alpha = 3 - 4C_{12}^\alpha/(C_{11}^\alpha + C_{12}^\alpha)$$

Here and henceforth the subscript denotes which component of the composite the symbol refers to: "1" indicates it refers to the binding matrix and "2" indicates that it refers to the material of the fibre; the square brackets indicate a jump in the value of the quantity contained within them $[F] = F^1 - F^2$.

For problems $I_{\alpha 3}$, J_α ($\alpha = 1, 2$) the complex potentials have the form (3.1). In this case, in the expression for the functions $\psi_1(z)$ the last sum must be equated to zero. Unlike problems $I_{\beta\beta}$, I_{12}, J_3, these problems are connected (the non-zero functions S and Φ are expressed in terms of the potential $\psi_1(z)$). From the satisfaction of the matching conditions at the interface between the fibre and the binding material we can set up an infinite system of algebraic equations for determining the complex coefficients a_k and b_k. From the coefficients a_1 and b_1 obtained from the infinite system we find the corresponding effective characteristics of the composite from the formulas

$$
\begin{gathered}
C_{55}^*=C_{44}'-\gamma \operatorname{Re} B_1,\ C_{45}^*=-\gamma \operatorname{Im} B_1 \\
e_{15}^*=e_{15}'-\gamma \operatorname{Re} B_2,\ e_{25}^*=-\gamma \operatorname{Im} B_2 \\
C_{44}^*=C_{44}'-\gamma \operatorname{Im} B_1,\ e_{14}^*=-\gamma \operatorname{Re} B_2 \\
e_{24}^*=e_{15}'-\gamma \operatorname{Im} B_2,\ \mathscr{E}_{11}^*=\mathscr{E}_{11}'+\gamma \operatorname{Re} B_2, \\
\mathscr{E}_{21}^*=\gamma \operatorname{Im} B_2,\ \mathscr{E}_{22}^*=\text{Э}_{11}'+\gamma \operatorname{Im} B_2 \\
B_1=2(C_{44}'a_1+e_{15}'b_1)/R,\ B_2=2(e_{15}'a_1-\mathscr{E}_{11}'b_1)/R
\end{gathered}
\tag{3.3}
$$

Note that the constants a_1 and C_1 in (3.2) and a_1 and b_1 in (3.3) vary depending on the type of problem being solved.

From the complete matrix of the effective moduli C_{ij}^*, e_{kl}^*, $\mathscr{E}_{mn}^*$ obtained we can determine any other set of parameters representing the piezoactive medium, in particular the piezoelectric moduli d_{km}^*, the volume piezoelectric modulus $d_V^* = d_{33}^* + 2d_{31}^*$, the piezoelectric sensitivity $g_V^* = d_V^*/\mathscr{E}_{33}$ [12], etc.

4. We will present some results of calculations to determine the effective characteristics of the piezoelectric composites considered.

In Fig. 2 the continuous curves represent the effective characteristics normalized to the corresponding values for a piezoactive composite in the case of a porous composite of type 3–1 with a hexagonal structure as a function of the volume concentration of pores γ. We chose PZT-4 piezoelectric ceramics [12] as the piezoactive matrix.

The characteristic features of a piezoelectric composite of this kind are the fact that the piezoelectric moduli d_{33}^* and d_V^* (curves 1 and 2) are practically independent of the volume concentration of pores γ, and the piezoelectric sensitivity g_V^* increases sharply as γ increases (curve 3).

Note that the filling of the pores with epoxy resin (Young's modulus $E = 3$ GPa, Poisson's ratio $\nu = 0.4$ and permittivity $\mathscr{E}/\mathscr{E}_0 = 5$) does not lead to any qualitative change in this behaviour, and only the quantitative features change slightly.

The dashed curves in Fig. 2 correspond to the effective values for a piezoelectric composite of type 1–3, which is a fibre of PZT-4 piezoelectric ceramics in a filler of epoxy resin (hexagonal packing). Along the abscissa we have plotted the volume concentration of piezoactive fibres γ.

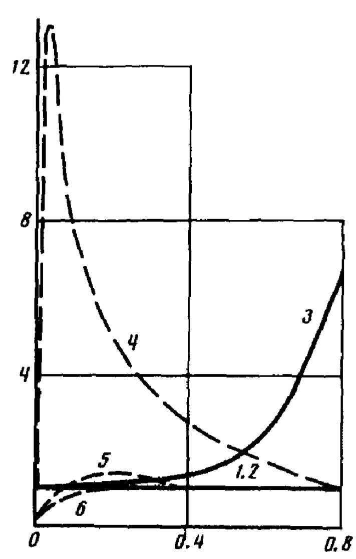

FIG. 2.

Typical features of such a composite are the considerable increase in the piezoelectric sensitivity g_V^* (curve 4) for small concentrations of piezoelectric ceramic fibres, the presence of a local maximum in the value of the volume piezoelectric modulus d_V^* (curve 5) and a slight dependence of the piezoelectric modulus d_{33}^* (curve 6) on the parameter ν over a wide range.

In addition to calculations of the properties of composites with hexagonal packing we carried out calculations for square packing. Note that in both cases the symmetry of the effective properties of the composite is identical with the symmetry of the piezoactive component. An exception is the modulus of elasticity C_{66}^* for a square structure, which is not equal to $(C_{11}^* - C_{12}^*)/2$. It was established from calculations that for small volume concentrations of the fibres (pores) the effective moduli in both cases are practically identical. The difference between them becomes considerable, however, as the volume concentration of the fibres increases. For example, for the moduli C_{11}^*, C_{13}^*, C_{44}^*, e_{31}^*, e_{15}^* for $\gamma = 0.75$ the difference exceeds 25%.

In conclusion we give some formulas for determining the effective properties of the fibre piezoelectric composites considered. The relative error

$$\Delta = 100\% \cdot (m - m_0)/m \tag{3.4}$$

for these formulas, which follows from the analytical solution of problems in a periodicity cell does not exceed 1% over a range of variation of γ from 0 to 0.4. In (3.4) m is the accurate value of the modulus and m_0 is the approximate value calculated from formulas which, for example, for the moduli C_{55}^*, e_{15}^* and $\mathscr{E}_{11}^*$ have the form

$$C_{55}^* = C_{44}^1 - 2\gamma C_{44}^1 \beta_1 - 2\gamma e_{15}^1 \beta_2$$
$$e_{15}^* = e_{15}^1 - 2\gamma e_{15}^1 \beta_1 + 2\gamma \mathscr{E}_{11}^1 \beta_2$$
$$\mathscr{E}_{11}^* = \mathscr{E}_{11}^1 + 2\gamma e_{15}^1 \beta_1 - 2\gamma \mathscr{E}_{11}^1 \beta_2$$

where β_1, β_2 satisfy the system of equations

$$a_{11}\beta_1 + a_{12}\beta_2 = a_{10},$$
$$a_{21}\beta_1 + a_{22}\beta_2 = a_{20}$$

in which

$$a_{11} = C_{44}^1(1+\gamma) + C_{44}^2(1-\gamma)$$
$$a_{12} = a_{21} = e_{15}^1(1+\gamma) + e_{15}^2(1-\gamma)$$
$$a_{22} = -\mathscr{E}_{11}^1(1+\gamma) - \mathscr{E}_{11}^2(1-\gamma)$$

The coefficients a_{10} and a_{20} for determining C_{55}^*, e_{15}^* have the form

$$a_{10} = C_{44}^1 - C_{44}^2, \quad a_{20} = e_{15}^1 - e_{15}^2$$

while for determining $\mathscr{E}_{11}^*$

$$a_{10} = e_{15}^1 - e_{15}^2, \quad a_{20} = \mathscr{E}_{11}^2 - \mathscr{E}_{11}^1$$

Other independent moduli can be found from the formulas:

$$C_{11}^* = \langle C_{11} \rangle - \gamma p_1(1)\alpha_1 - \gamma p_2(1)\alpha_2$$
$$C_{21}^* = \langle C_{21} \rangle + \gamma p_1(1)\alpha_1 - \gamma p_2(1)\alpha_2$$
$$C_{31}^* = \langle C_{31} \rangle - \gamma [C_{31}]\alpha_2, \quad e_{31}^* = \langle e_{31} \rangle - \gamma [e_{31}]\alpha_2$$
$$C_{33}^* = \langle C_{33} \rangle - \gamma [C_{31}]^2\alpha_3, \quad \mathscr{E}_{33}^* = \langle \mathscr{E}_{33} \rangle + \gamma [e_{31}]^2\alpha_3$$
$$e_{33}^* = \langle e_{33} \rangle - \gamma [C_{31}][e_{31}]\alpha_3$$
$$C_{66}^* = C_{66}^1 - \gamma [C_{66}](1+\varkappa_1)/(1+\varkappa_1\varkappa+\alpha_{33})$$
$$\alpha_1 = 1 - (1+\varkappa_1)/(1+\varkappa_1\varkappa+\alpha_{11})$$
$$\alpha_2 = (1-\gamma)(\varkappa_2-1)p_2(1)/(C_{66}^1\alpha_{22})$$
$$\alpha_{11} = (1-\varkappa)\{(\varkappa_1 + 5S_4/\pi^2)\gamma - 6S_4R^4\}$$
$$\alpha_{22} = 2\varkappa(1-\gamma) + (\varkappa_2-1)(1+2\gamma/(\varkappa_1-1))$$
$$\alpha_{33} = (1-\varkappa)\{(\varkappa_1 - 5S_4/\pi^2)\gamma + 6S_4R^4\}$$
$$\alpha_3 = \alpha_2/p_2(1), \quad \varkappa = C_{66}^2/C_{66}^1$$

The parameter $S_4 = 3.151212$ for a square structure and $S_4 = 0$ for a hexagonal structure.

The results obtained enable us to predict the effective properties of a unidirectional fibre composite with a periodic structure depending on the elastic, electric and geometrical parameters of its components.

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